Journal of Statistical Physics, Vol. 26, No. 1, 1981

Ground State Degeneracy and Ferromagnetism in a Spin Glass

J. E. Avron,[1,2] G. Roepstorff,[1,3] and L. S. Schulman[1,4]

Received September 24, 1980

We prove three results for the two-dimensional Ising spin glass model on a square lattice: (a) finite entropy for the ground state; (b) ferromagnetism for low concentrations of antiferromagnetic bonds and low temperatures; (c) vanishing magnetization for a spin glass with equal concentrations of ferro- and antiferromagnetic bonds.

KEY WORDS: Spin glass; ferromagnetism; ground state degeneracy; Peierls argument, frustration.

1. INTRODUCTION

The Ising spin glass model on $\mathbb{Z}^2$ has spins $\sigma_i = \pm 1$ on sites $i \in \mathbb{Z}^2$ and nearest-neighbor interaction with coupling $J_b = \pm 1$, $b \in B$, B the set of bonds. The J_b are independent random variables with $\mathrm{Prob}(J_b = -1) = x$. Thermodynamic functions such as the free energy and the magnetization are random variables on the space of bond configurations.

Consider a finite volume Λ and let $\sigma \in \{-1, 1\}^{\Lambda}$ and $J \in \Omega = \{-1, 1\}^{B(\Lambda)}$ denote the spin and bond configurations, respectively, where $B(\Lambda) = \{b \mid \partial b \subset \Lambda\}$. The Hamiltonian is defined as

$$H_{\Lambda}(J, \sigma) = -\sum_{b \in B(\Lambda)} J_b \sigma_{\partial b} \tag{1.1}$$

[1] Physics Department, Technion, Haifa, Israel.

[2] Present address: Division of Mathematics, Caltech, Pasadena, California.

[3] Present address: Institute for Advanced Study, Princeton, New Jersey. Permanent address: Physics Department, RWTH Aachen, Federal Republic of Germany.

[4] Address for 1980–81: IBM Research Center, P.O. Box 218, Yorktown Heights, New York 10598.

0022-4715/81/0900-0025$03.00/0

where

$$\sigma_A = \prod_{i \in A} \sigma_i \qquad \text{for } A \subset \Lambda$$

The free energy per site is

$$f_\Lambda(J) = \frac{-1}{|\Lambda| \beta} \ln Z_\Lambda(J) \tag{1.2}$$

where ($\beta = 1/kT$)

$$Z_\Lambda(J) = \sum_\sigma e^{-\beta H_\Lambda(J,\sigma)} \tag{1.3}$$

The magnetization is defined as

$$m_\Lambda(J) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle \sigma_i \rangle_\Lambda(J) \tag{1.4}$$

$$\langle \sigma_i \rangle_\Lambda(J) = Z_\Lambda(J)^{-1} \sum_\sigma \sigma_i e^{-\beta H_\Lambda(J,\sigma)} \tag{1.5}$$

Finally, σ^0 is a ground state if, for all σ satisfying some specified boundary conditions,

$$H_\Lambda(J, \sigma) \geqslant H_\Lambda(J, \sigma^0) \tag{1.6}$$

Let $G_\Lambda(J)$ be the number of ground states. Then the ground state entropy is defined to be

$$S_\Lambda(J) = \frac{1}{|\Lambda|} \ln G_\Lambda(J) \tag{1.7}$$

Let $\mu_x(J)$ denote the product measure on Ω,

$$\mu_x(J) = x^{N^-} (1-x)^{N^+} \tag{1.8}$$

where N^+ is the number of bonds $b \in B(\Lambda)$ such that $J_b = \pm 1$. We write

$$\overline{(\cdot)}_x = \sum_J \mu_x(J)(\cdot) \tag{1.9}$$

for the average of some quantity with respect to J.

Standard arguments assure the existence of the thermodynamic limit, under suitable conditions on the growth of Λ, for the free energy and the ground state entropy.[1,4] Obviously, the limits are invariant with respect to the translation group τ, and (Ω, μ_x, τ) with $\Lambda = \mathbb{Z}^2$ is a Bernoulli system. It is then a consequence of the ergodic theorem that μ_x-almost everywhere

$$\lim_{\Lambda \uparrow} S_\Lambda(J) = \lim_{\Lambda \uparrow} \overline{(S_\Lambda)}_x \tag{1.10}$$

A similar formula holds for $f_\Lambda(J)$ and $m_\Lambda(J)$. Our main results are the following:

Theorem 1.1. For all Λ (sufficiently large) and all boundary conditions

$$\overline{(S_\Lambda)}_x \geqslant \frac{p \ln 3}{100}$$
$$p = 132\left[x^{16}(1-x)^{120} + x^{120}(1-x)^{16}\right]$$

The theorem guarantees finite zero-point entropy for $0 < x < 1$.

Remark. An alternative definition of the residual entropy is the $T \to 0$ limit of the finite temperature entropy. The two definitions need not always coincide. For a study of this question see Ref. 1.

Theorem 1.2. Let $c = [3(3+\sqrt{2})]^{-1/2} \simeq 0.274$ and suppose that

(1) $$T = 0, \qquad 0 \leqslant x \leqslant (c/2)^2 \simeq 0.0188$$

or that

(2) $$x = 0, \qquad 0 \leqslant 2kT < \ln(\sqrt{2}+1)$$

or that

(3) $$2e^{-2\beta} + 2^{5/4}x^{1/2} \leqslant c$$

Then, for positive boundary conditions and some $m_0 > 0$ independent of Λ, $\overline{(m_\Lambda)}_x \geqslant m_0$.

The theorem guarantees the existence of a ferromagnetic phase. Under hypothesis (2) the result is well-known. With hypotheses 1 or 3, the proof is based on the Peierls–Griffiths argument and features colored contours.

Remarks. (1) Cohomology is in many ways the natural language to describe the Ising spin glass. In this paper we have avoided its use so as to bring the physics to the fore. This is the reason for treating only the two-dimensional system. In dimensions $n \geqslant 3$, the Peierls argument also guarantees ferromagnetism. Moreover, using techniques of Ref. 11, sharper estimates may be obtained for $n = 2$: the constant 0.0188 in hypothesis (1) of the theorem may be replaced by 0.0283. This will be discussed elsewhere.

(2) The phase diagram of the Ising spin glass on a square lattice is symmetric under $x \to 1 - x$. This establishes an antiferromagnetic phase for $1 - x$ and T small. The symmetry follows from a global gauge transformation that flips all bonds and every second spin.

Theorem 1.3. For all $A \subset \Lambda$

$$\left(\overline{\langle \sigma_A \rangle_\Lambda}\right)_{x=1/2} = 0$$

independent of the boundary conditions.

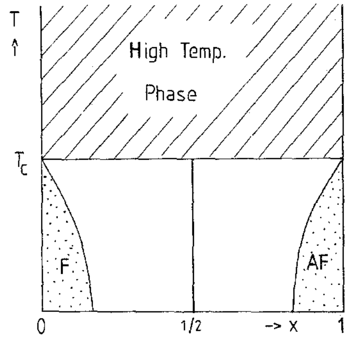

Fig. 1. Regions in the (x, T) plane for which there are rigorous results (schematic) for the spin glass. The ferromagnetic and antiferromagnetic phases due to Theorem 1.2 are dotted. The magnetization vanishes on the line $x = 1/2$ and in the hatched region $T > T_c$ representing the (unique) high-temperature phase as implied by the correlation inequalities (1.11). The symmetry of the phase diagram about $x = 1/2$ reflects the gauge symmetry.

The theorem precludes a ferromagnetic phase for $x = 1/2$. It does not, however, preclude other sorts of order. In particular, it says nothing about a spin glass phase.

These theorems and the fact[7] that

$$|\langle\sigma_A\rangle_\Lambda(J)| \leqslant \langle\sigma_A\rangle_\Lambda\,(J = +1) \tag{1.11}$$

which states that the critical temperature for the Ising ferromagnet is an upper bound on the critical temperature for the spin glass, provide rigorous information on the phase diagram of the Ising spin glass as shown in Fig. 1.

2. COLORED CONTOURS

Ferromagnetic Ising spin systems with $J_b = 1$ are equivalent to contour models on the dual lattice.[3] For two-valued bond variables $J_b = \pm 1$ we introduce colored contours.

Definition 2.1. A g^* string is a maximal, connected chain of bonds in the dual lattice Λ^* such that $J_b = -1$ for all $b^* \in g^* \subset B(\Lambda^*)$. When two g^* strings cross, the ambiguity at crossing is resolved in the usual way.[3]

Remarks. (1) g stands for green. g^* strings characterize the antiferromagnetic bonds in the lattice and are independent of the spin configuration. See Fig. 2.

(2) Another model where g^* strings arise is the random, dilute magnet.[4] In this model $J_b \in \{0, 1\}$, and Definition 2.1 should be modified by replacing -1 by 0.

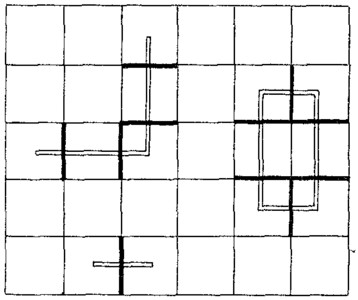

Fig. 2. A bond configuration J and the g^* strings (double lines) associated with it. Antiferromagnetic bonds are represented by heavy lines.

(3) $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Objects in the dual lattice are marked with an asterisk. An asterisk also denotes the operation that identifies a plaquette in the dual lattice with a site in Λ, a bond in Λ^* with a bond in Λ, and a site in Λ^* with a plaquette in Λ. Thus for $k = (k_1, k_2) \in \Lambda$ we have $k^* = (k_1 + \frac{1}{2}, k_2 + \frac{1}{2}) \in \Lambda^*$.

Definition 2.2. An r^* string is a maximal, connected chain of bonds in the dual lattice Λ^* such that $J_b \sigma_{\partial b} = -1$ for all $b^* \in r^* \subset B(\Lambda^*)$. See Fig. 3.

Remarks. (1) r stands for red. The r^* strings describe the unsatisfied bonds. At high temperatures red strings are long and winding, at low temperatures they are short and scarce.

(2) Up to an additive constant the energy of a configuration is the total length of the r^* strings.

We now recall the usual Peierls contours. These are contours, C^*, in the dual lattice that separate + spins from − spins. The Peierls contours

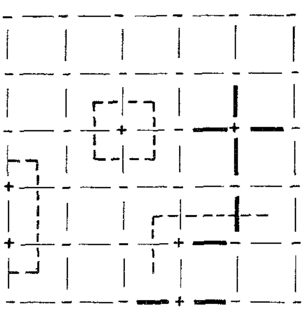

Fig. 3. A spin-bond configuration and the corresponding r^* strings (broken lines). Antiferromagnetic bonds are represented by heavy lines.

uniquely determine the spin configuration and are closed curves provided one uses positive boundary conditions.

Lemma 2.3. Let b^* be a dual bond on some Peierls contour C^*. Then b^* belongs either to some g^* string or to some r^* string but not to both.

Proof. By definition, $\sigma_{\partial b} = -1$ for every $b^* \in C^*$. If $J_b = +1$ (-1), then b^* belongs to an r^* string (g^* string).

The categories C^*, g^*, and r^* in the lemma may be permuted. Thus every bond is part of either zero or two kinds of string. The lemma also holds for the random magnet with $J_b \in \{0, 1\}$.

Let $P(\Lambda)$ be the set of plaquettes in Λ. A plaquette $p \in P(\Lambda)$ is said to be frustrated if

$$J_{\partial p} = \prod_{b \in \partial p} J_b = -1 \tag{2.1}$$

A bond configuration J is said to describe irrelevant disorder if there are no frustrated plaquettes. In particular, the pure ferromagnet ($J_b = +1$) and the antiferromagnet ($J_b = -1$) have no frustrated plaquettes.

Lemma 2.4. r^* strings and g^* strings have the frustrated plaquettes as common set of bounding dual sites. In detail, every $r^*(g^*)$ string connects two frustrated plaquettes, or is a closed contour, or connects the boundary of Λ to a frustrated plaquette or to the boundary itself.

Proof. If p is a frustrated plaquette, one or three of its bounding bonds are antiferromagnetic. Thus, precisely one g^* string terminates at the dual site p^*. Likewise, one or three bonds are dissatisfied. Thus, precisely one r^* string terminates at p^*.

3. GROUND STATE DEGENERACY

In the ground state, the total length of the r^* strings is minimal.[5,9] In particular, determination of the ground state degeneracy is a geometric problem. Roughly, the finite entropy is a consequence of two facts:

(1) There may exist different possible pairings of frustrated plaquettes by r^* strings with the same energy.

(2) Two frustrated plaquettes that do not lie on a common row or column have at least two shortest connecting r^* strings.

At first it may appear that finite entropy follows trivially from configurations of the type shown in Fig. 4a. In the ground state one expects the two plaquettes to be paired if they are far from the remaining frustrated plaquettes, and, as the two plaquettes may be connected by at least two

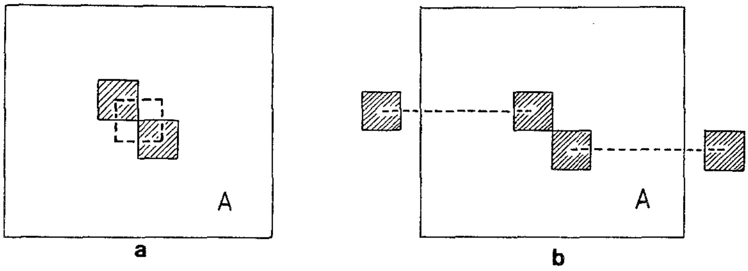

Fig. 4. (a) If there are only two frustrated plaquettes in A, they are likely to be paired by an r^* string. In the situation shown, there are two possibilities for choosing an r^* string of minimal length. (b) A situation is sketched where the two frustrated plaquettes in A are not connected by an r^* string. Here the region A does not contribute to the degeneracy of the ground state.

shortest r^* strings, the configuration leads to a twofold degeneracy, at least. This type of frustration has constant density in $\mathbb{Z}^2$. Thus, the degeneracy of the ground state in a large box Λ increases exponentially with $|\Lambda|$ leading to finite entropy. However, this argument is incomplete. The two plaquettes may not be paired at all in the ground state, even if they are well isolated from the remaining frustrated plaquettes, as is the case in Fig. 4b, where the degeneracy is lost. Thus, a diagonal pair of frustrated plaquettes may or may not lead to a degenerate ground state depending on the boundary conditions.

The problem then is to show that there exists a configuration $\varphi(J)$ of frustrated plaquettes in $A \subset \Lambda$ (for A sufficiently large) such that $H_A(J, \sigma)$ has a degenerate ground state *independent of the frustration in* $\Lambda \backslash A$ *for all* Λ.

Notice that, in the ground state, two r^* strings cannot run parallel to each other for long stretches. The reason is illustrated in Fig. 5. More precisely, let r_1^*, and r_2^* be ground state strings parallel to some axis and let $\pi r_{1,2}^*$ be the projection onto this axis. Then

$$\operatorname{dist}(r_1^*, r_2^*) \geqslant |\pi r_1^* \cap \pi r_2^*| \tag{3.1}$$

Nondegeneracy is a strong condition on the geometry of the r^* strings. In

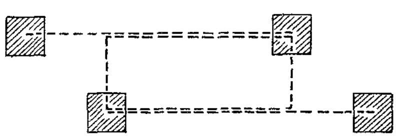

Fig. 5. Two r^* strings of a ground state never run parallel for large distances, since the total length of such strings may be reduced by adding a closed contour and thereby eliminating the double line sections.

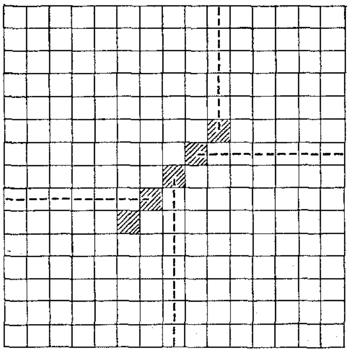

Fig. 6. The configuration φ of five frustrated plaquettes in A leads to a degenerate ground state independent of the frustration in $\Lambda \backslash A$. Up to four r^* strings may be drawn causing no degeneracy.

fact, a ground state in A is nondegenerate only if (a) all r^* strings in A are straight lines, and (b) no two r^* strings intersect.

We shall now construct a configuration φ of frustrated plaquettes such that (3.1) and (a) and (b) cannot simultaneously hold. This then implies that the ground state in A is degenerate for all boundary conditions. φ is shown in Fig. 6. It has five frustrated plaquettes on the diagonal in A. If the ground state were nondegenerate, all of the five plaquettes would be connected to the boundary of A by straight r^* strings. By virtue of (3.1) no two strings run in the same direction. Thus, once the four directions of the compass have been exhausted, the string of at least one plaquette must violate (a) or (b). As a consequence, $G_A(J) \geqslant 2$ independent of the frustration in $\Lambda \backslash A$. An improved lower bound on the entropy follows from configurations $\varphi'(J')$ in $A' \subset \Lambda$ with five frustrated plaquettes as shown in Fig. 7. Using the same argument we find

$$G_{A'}(J') \geqslant 3 \tag{3.2}$$

The probability for φ' to occur is

$$\mathrm{Prob}(\varphi') \geqslant p = 132\left[x^{16}(1-x)^{120} + x^{120}(1-x)^{16}\right] \tag{3.3}$$

To complete the argument, consider Λ containing n squares $A_j'(j = 1, \ldots, n)$ separated by unit corridors. The bonds in distinct squares are independent random variables. The probability that precisely m out of n squares have the configuration φ' is

$$p_n(m) = \binom{n}{m} p^m (1-p)^{n-m} \tag{3.4}$$

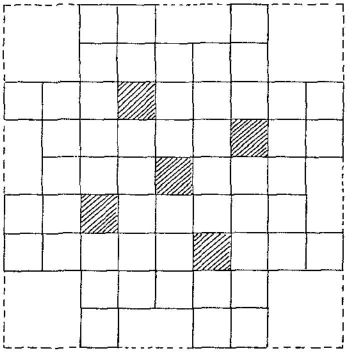

Fig. 7. The region A' and the configuration of five frustrated plaquettes, φ', yield an improved estimate of the entropy.

Consequently,

$$\overline{(S_\Lambda)}_x = \frac{1}{|\Lambda|} \sum_J \mu_x(J) \ln G_\Lambda(J)$$
$$\geqslant \frac{1}{|\Lambda|} \sum_{m=0}^{n} p_n(m) m \ln 3 = \frac{n}{|\Lambda|} p \ln 3 \tag{3.5}$$

For all n, $|\Lambda|/n < |A'| = 100$. The bound (3.5) is poor. For $x \to 0$ one expects $\bar{S} = 6x^2 \ln 2 + O(x^3)$.

4. THE PEIERLS ARGUMENT

Choosing positive boundary conditions we know that all C^* contours that separate plus from minus spins are closed. Consider a C^* contour, j, with length l. Its energy is

$$E_n = l - n + \epsilon n \tag{4.1}$$

where n is the number of dual bonds $b^* \in j$ such that $J_b = \epsilon$. We mainly deal with $\epsilon = -1$ (spin glass) but shall also be interested in the case $\epsilon = 0$ (random magnet). Flipping the spins inside j gives a spin configuration with energy lower by $2E_n$. Let $\chi_j(\sigma)$ denote the characteristic function of the contour j. Then[3]

$$\langle \chi_j \rangle_\Lambda(J) \leqslant (1 + e^{2\beta E_n})^{-1} \tag{4.2}$$

The number of contours of length l, N_l, is bounded[3] by

$$N_l \leqslant |\Lambda| 4 \times 3^{l-2}/l \tag{4.3}$$

Suppose

$$\overline{(\langle\chi_j\rangle_\Lambda)}_x \leqslant \sum_{n=0}^{l}\binom{l}{n}x^n(1-x)^{l-n}\left(1+e^{2\beta E_n}\right)^{-1} \leqslant y^l \tag{4.4}$$

Then the fraction of minus spins is bounded from above by

$$Q = \frac{1}{|\Lambda|}\sum_{l=4,6,8\ldots}\left(\frac{l}{4}\right)^2 N_l y^l \leqslant \frac{1}{18}\left[\frac{(3y)^2}{1-(3y)^2}\right]^2\left[2-(3y)^2\right] \leqslant \left(\frac{3y^2}{1-9y^2}\right)^2 \tag{4.5}$$

If

$$y < c = \left[3(3+\sqrt{2})\right]^{-1/2} \simeq 0.2748 \tag{4.6}$$

then $Q < 1/2$ and $\overline{(m_\Lambda)}_x > 0$ for all Λ. Explicit estimates for y are

(1) $$y = 2x^{1/2}, \qquad T = 0, \qquad \epsilon = -1 \tag{4.7}$$

(2) $$y = x, \qquad T = 0, \qquad \epsilon = 0 \tag{4.8}$$

(3) $$y = e^{-2\beta}, \qquad x = 0 \tag{4.9}$$

(4) $$y = 2e^{-2\beta} + 2^{5/4}x^{1/2}, \qquad \epsilon = -1 \tag{4.10}$$

To derive (4.10) we used

$$\begin{aligned}\sum_{n=0}^{l}\binom{l}{n}x^n(1-x)^{l-n}\left(1+e^{2\beta(l-2n)}\right)^{-1} &\leqslant \sum_{n=0}^{l/2}\binom{l}{n}\left[x^n e^{-2\beta(l-2n)} + x^{l-n}\right] \\ &\leqslant (2e^{-2\beta})^l + 2(2x^{1/2})^l \\ &\leqslant (2e^{-2\beta} + 2^{5/4}x^{1/2})^l \qquad (l \geqslant 4)\end{aligned}$$

Remarks. (1) The best estimates available of critical concentration x_c (at $T = 0$) are based on Monte-Carlo calculations and give $x_c \simeq 0.12 \pm 0.04$.[8]

(2) For the random magnet ($\epsilon = 0$), the Peierls argument predicts ferromagnetism for $T = 0$ and $0 \leqslant x < c \simeq 0.27$. Actually, it is believed that the threshold is $x_c = 0.5$,[4] coinciding with the bond percolation threshold. This means that the present argument underestimates x_c by a factor of about 2 though it improves previous estimates.[4]

5. GAUGE INVARIANCE

Spin glasses may be viewed as lattice gauge theories with external random gauge fields. In gauge theories it is known that local gauge invariance implies zero magnetization. Similar results are expected to hold for certain spin glasses. This is the basic idea behind our proof of Theorem 3.

We start out by introducing the coboundary operator $\hat{\partial}$ formally defined by $\hat{\partial} c = (\partial c^*)^*$, where c may be a set of sites or bonds. For instance, the coboundary of a site i is the set of four bonds having i as common vertex. A gauge transformation at i flips σ_i and J_b with $b \in \hat{\partial} i$. It induces a map of the variables $\sigma_A (A \subset \Lambda)$ and $J_V[V \subset B(\Lambda)]$. More generally, a gauge transformation g_C flipping all spins inside $C \subset \Lambda$ has the following effect on the spin (bond) configuration:

$$\begin{aligned} g_C \sigma_A &= (-1)^{|C \cap A|} \sigma_A \\ g_C J_V &= (-1)^{|V \cap \hat{\partial} C|} J_V \end{aligned} \tag{5.1}$$

Local gauge transformations leave the energy $H_\Lambda(J, \sigma)$ invariant but change the measure $\mu_x(J)$ to $\mu_x(g_C J)$. By (1.8), $\mu_{1/2}(J)$ is gauge invariant. Let $A \subset \Lambda$ and $g = g_C$ with $|C \cap A| = \text{odd}$. Consider the function

$$\phi(J) = \langle \sigma_A \rangle_\Lambda(J) \tag{5.2}$$

Then

$$\begin{aligned} \phi(gJ) &= Z_\Lambda(gJ)^{-1} \sum_\sigma \sigma_A e^{-\beta H_\Lambda(gJ,\sigma)} \\ &= Z_\Lambda(J)^{-1} \sum_\sigma g\sigma_A e^{-\beta H_\Lambda(J,\sigma)} = -\phi(J) \end{aligned} \tag{5.3}$$

where the gauge invariance of H_Λ and (5.1) was used. However, by the invariance of $\mu_{1/2}$,

$$\begin{aligned} 0 &= \sum_J \mu_{1/2}(J)\left[\phi(J) + \phi(gJ)\right] \\ &= 2\sum_J \mu_{1/2}(J)\phi(J) = 2\left(\overline{\langle \sigma_A \rangle_\Lambda}\right)_x \end{aligned}$$

Remarks. (1) This is essentially the argument of the Elitzur theorem.[6] See also Ref. 10.

(2) The spin glass phase is defined by the Edwards–Anderson order parameter $(\overline{\langle \sigma_0 \rangle^2})_x$. Since this parameter is gauge invariant, our arguments do not apply here.

ACKNOWLEDGMENTS

This work is partially supported by the US–Israel Binational Science Foundation, grant No. 1573. G.R. acknowledges the hospitality of the physics department of the Technion and J.A. of the FU, Berlin and of CPT II, Marseille. We wish to thank Professor C. M. Newman for useful discussions.

REFERENCES

1. M. Aizenman and E. H. Lieb, *J. Stat. Phys.* to appear.
2. E. Fradkin, B. H. Huberman, and S. H. Shenker, *Phys. Rev. B* **18**:4789 (1978).
3. R. B. Griffiths, *Phys. Rev.* **136**:A437 (1964).
4. R. B. Griffiths and J. L. Lebowitz, *J. Math. Phys.* **9**:1284 (1968).
5. S. Kirkpatrick, *Phys. Rev. B* **16**:4630 (1977).
6. J. Kogut, *Rev. Mod. Phys.* **51**:659 (1979).
7. J. L. Lebowitz, *J. Stat. Phys.* **16**:463 (1977).
8. I. Morgenstern and K. Binder, *Phys. Rev. B* **22**:288 (1980).
9. G. Toulouse, *Commun. Phys.* **2**:115 (1977).
10. F. Wegner, *J. Math. Phys.* **12**:2259 (1971).
11. C. Y. Weng, R. B. Griffiths, and M. E. Fisher, *Phys. Rev.* **162**:475 (1967).